MAKING GENERALIZATIONS EXPLICIT: AN INFERENTIAL PERSPECTIVE ON CONCEPT-FORMATION OF VARIABLES

Florian Schacht, Stephan Hußmann

TU Dortmund University, Mathematics Education, Germany

The transition from preformal and propaedeutic generalization-actions to a symbolically explicit use of the concept of variable has been a matter of significant attention in mathematics education, for example in the context of generalization processes on a preformal level and regarding the specific nature of algebraic concepts. This contribution offers an inferential theoretical perspective to study the relation of geometric and arithmetic notions when dealing with figural growing patterns and arithmetic sequences. By reconstructing "individual commitments" we show results of the impacts of this relation on the individual notion and explicit use of the concept of variable. Finally, the results also show that the concept of "propaedeutics" itself gains an extension in the light of the theoretical framework.

GENERALIZING IN THE CONTEXT OF VARIABLE

The concept of variable in algebraic expressions is one of the most important mathematical objects within mathematics classroom (Kieran, 2007, Usiskin, 1988, Lee, 1996). Its fundamental role is due to its conceptual nature as a tool to make explicit patterns and structures in form of generalizations. This paper presents results of a study (Schacht, 2012, Young Researchers Promotion Award 2012 by the GDM) on the propaedeutic and then symbolically explicit use of the concept of variable in early algebra. The explication of the concept of variable in mathematics classroom (usually in 5^{th} grade) does not mark the beginning of algebraic experiences for the students. These experiences are often deeply rooted within the conceptual dimensions of dealing with patterns, a functional context, equivalence and equations on a preformal level (e.g. Cooper & Warren, 2011). Cooper & Warren especially point out the importance of generalizing for learning algebra: "improving one's ability to generalise lies at the foundation of efforts to enhance participation in and learning of algebra." (Cooper & Warren, 2011, p. 190) There are many productive examples that use this idea within the context of early algebra (cf. Lee, 1996). For example, Lee (1996) points out to start early with generalization-actions: "Generalization is one of the important things we "do" in algebra and therefore something students should be initiated into fairly early on." (Lee, 1996, p. 103) There are two different types of generalization, a recursive and an explicit one (e.g. Lannin, 2005), whereas "young students develop these abilities from recursive to explicit" (Cooper & Warren, 2011, p. 197). With variables in algebraic expressions, students can make these generalizations explicit, for example in dynamic figural growing patterns or in arithmetic sequences.

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It is important to note, that the notion of generalizing can both be done on a geometric level (with figural growing patterns) and on an arithmetic level, whereas the geometric level can be seen as essential in order to build a solid foundation for the future understanding of functional contexts (Moss & London McNab, 2011, p. 297). Becker and Rivera (2011, p. 363) even point out that the ability to generalize figural growing patterns supports the conceptualization of the concept of variable significantly.

Within our study, a learning environment was used (Hußmann et al., 2012) that is mainly influenced by these ideas and that follows the following aspects: First, students use numerical expressions to describe static figural patterns (e.g. $2\cdot3+6\cdot5$), then they use expressions like these to describe the number of dots in figural growing patterns (e.g. for the linear sequence $2+1\cdot5$, $2+2\cdot5$, $2+3\cdot5$, $\cdots$). Then, in order determine the number of dots in for example the 1000th element of the sequence, they use this linear structure to determine $2+1000\cdot3$. Finally, they use the algebraic expressions with variables to make the generalization of the pattern explicit.

Within this learning environment, the concept of variable is used to both determine the number of dots for different elements of a given sequence and pattern and to describe the structure of a given (figural or arithmetic) sequence. The potential of dealing with patterns and describing can be seen in the specific nature of patterns:

> Mathematical visualization and growing patterns (…) can mediate between the mathematical structure and the student's thinking because of their special ‚double nature' (they are on the one hand concrete objects, which can be dealt with, which can be pointed at and counted, (…) and at the same time they are symbolic representatives of abstract mathematical ideas). (Böttinger, et al., 2009, p. 151)

Sfard et al. (1994) point out a similar notion the dual nature of algebraic concepts: "the operational outlook in algebra is fundamental and the structural approach does not develop immediately" (p. 209). Regarding the variable, this dichotomy means that it can be used specifically as a tool for example to find out the number of dots in a certain element of a growing pattern. At the same time the variable can be used as structural objects in algebraic expressions with variables for example to describe mathematical growing patterns.

These insights refer to the different epistemological statuses of mathematical concepts and especially to the variable itself. Our results suggest, that these insights for algebraic concepts can be extended to a propaedeutic understanding. Within the context of the study and within the processes of concept-formation toward the variable, the students were engaged in many generalization processes when dealing with geometric and arithmetic sequences without using the concept of variable explicitly. Regarding these processes on a micro-level, there is a need for research concerning the question, how far these geometric and arithmetic generalization processes correspond to the (individual) later notion of the concept of variable as a tool or as a theoretical object.

THEORETICAL FRAMEWORK, RESEARCH QUESTIONS AND DESIGN

The inferentialist perspective

In order to describe both individual processes of concept formation as well as the process of constructing learning environments and in order to structure the subject matter we developed a consistent theoretical framework. Its foundations pick up philosophical ideas of Kant, Frege, Wittgenstein, Heidegger and Frege. Also, the theory of inferentialism (Brandom, 1994) was adopted to develop this framework.

Within this framework, commitments are seen as (reconstructed) assertions in a propositional form, that the individual student acknowledges and holds to be true. Commitments can be made explicit. In this perspective it is one of the central background-theoretical assumptions, that doing mathematics a highly social process: "At the core of discursive practice is the game of giving and asking for reasons." (Brandom, 1994, p. 159) This basic theoretical assumption is used within a qualitative psychological research design. *Individual commitments* are seen as the smallest units of thinking and acting, that do not necessarily have to be true, but which are being held to be true by the individual. Within the interpretative process, these individual commitments are reconstructed turn-by-turn. Commitments can be acknowledged by the individual and they can be attributed to our discursive partners. As reasons in argumentation, individual commitments are inferentially related. That does not mean an inferential relation in the sense of classical logic though: inferential relations between two commitments do not have to be true or false, but they are held to be true or false by the individual!

In this commitment-based theoretical perspective, the notion of individual concept-formation is being extended: individual concept-formation is modeled as the development of individual commitments, that underlie our use of concepts (c.f. Schacht, 2012). Since our concepts are always inferentially related by the commitments we acknowledge, this implies a holistic perspective on concepts themselves: "One immediate consequence of such an inferential demarcation of the conceptual is that one must have many concepts in order to have any. For grasping a concept involves mastering the proprieties of inferential moves that connect it to many other concepts (...). One cannot have just one concept." (Brandom, 1994, p. 89) This fundamental insight is one of the foundations of the analysis, since the theoretical framework gives respect to the many concepts that are involved when learning the concept of variable for example.

Finally, this theoretical perspective has an important consequence for the notion of propaedeutics of concepts. Even if a certain concept is not yet symbolically explicit, there may be still identified a variety of individual commitments, that refer to the concept implicitly or explicitly. This means that even before the explication of the concept of variable it might be possible to reconstruct individual commitments that refer to the concept of variable for example in the context of generalizing arithmetic or

geometric patterns. This way, the genesis of individual concepts can be reconstructed and described within a fine-grained analysis (cf. Schacht, 2012, Hußmann et al., 2009).

Research Questions and Design

The inferential theoretical framework offers potential especially regarding its conceptual foundations. First the *inferential notion* of the framework faces an important nature of concepts: we never learn only one isolated concept (e.g. the concept of variable), but furthermore many other concepts, that may get a different and new shape within the practice, operations and new situations. The explication of the variable in mathematics classroom, that follows the exploration of geometric and arithmetic patterns, may have also influence on the concepts of *balance, equivalence or recursivity*. Within this perspective, the following research questions are posed:

- How do individual commitments, that refer to geometric and arithmetic concepts, influence the individual notion and use of the concept of variable?
- Within the process of concept-formation, how far do individual commitments relate, depending on i) the preformal and propaedeutic use within generalization processes and ii) the symbolically explicit use of the variable?

These research questions are posed within a qualitative research design. The case study (Schacht, 2012) was planned and conducted within the design-research project KOSIMA (see Barzel et al., 2013). About 60 students (11 years) worked within a learning-environment which introduces to the concept of variable. The theoretical framework was used to understand and describe individual processes of concept-formation by reconstructing individual commitments and their inferential relation. In this paper, the case *Orhan* is discussed in detail.

RESEARCH RESULTS AND DISCUSSION

Geometric and arithmetic commitments within generalization processes

The following scenes show the student Orhan dealing with linear dynamic figural patters with the arithmetic structure 2+6x. Within the learning process, Orhan has already dealt with static figural patterns. In the scenes below (Fig. 1), Orhan is given the figural pattern and he is asked to draw the next element of the sequence. He then counts the number of dots in the first three elements of the sequence, determines the arithmetic rule of the figural pattern (*add 6*, indicated by “2 + 6 = 8” and “8 + 6 = 14”, see Figure 1) and then he calculates the number of dots for the next element: 20 dots. Then he draws the next element with the shape of a triangle.

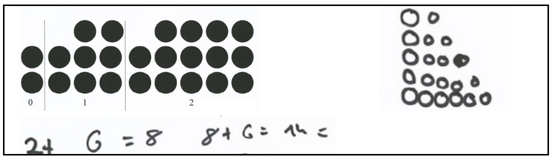

Figure 1: First three elements of a growing pattern. Orhan draws the fourth element.

Analyzing the transcript in detail, the reconstruction of the individual commitments shows that his actions in this scene are prototypical for many other scenes in a way, that they follow a certain scheme, that he repeats in different other scenes: Orhan first

counts the total number of dots of each element, he then determines the balance between each two elements and then generates the arithmetic rule that underlies the figural growing pattern. He then uses this arithmetic rule to determine the number of dots of the next element (see table 1 for a reconstruction of his commitments). He then continues the figural growing pattern by drawing the next element with a shape of a triangle. This is also a typical scene, because Orhan uses this shape in different scenes to continue differently structured growing patterns. By using the term "wall", he refers to the triangle-shaped structure that he often uses to continue the sequences geometrically.

1 Int.: Why did you draw the dots like this and not in a different way?

2 Orhan: Because, eh, the wall is easier for me.

The following table shows the reconstruction of Orhan's commitments within this scene. The arrow indicates the inferential relation, that means that a given commitment serves as a reason for the next one.

Commit-ment number	**Orhan's individual commitments (reconstruction)**	**inf. relation**	**geo. or arith.**
1	I determine the number of dots by counting them one-by-one	↙	geo.
2	I can find the arithmetic rule by determining the differences.	↙	arith.
3	The rule of the growing pattern is: you have to add 6 dots to the last element.	↙	arith.
4	I use the (arithmetic) rule of the pattern to determine the number of dots in the next pattern.	↙	arith.
5	The next element has 14+6=20 dots.	↙	arith.
6	I can use a triangle shape to visualize a given number of dots.		geo. / arith.

Table 1: Orhans individual commitments in an inferential structure

This reconstruction of Orhan's individual commitments shows a number of results. First, it shows the strong relation between commitments that refer to arithmetic (commitments number 2, 3, 4, 5 in Table 1) and to geometric concepts (1, 6 in Table 1). Also – typical for situations like these – Orhan initially changes from geometric commitments to arithmetic commitments when continuing a figural growing pattern. Originally, this task was posed in order to observe, if students see the geometric pattern (there is always a block of 6 dots added in 2 rows of 3 dots each) and if they use this pattern to continue the sequence. By doing so, we figured, they would initially use commitments that refer to geometric concepts (rows, geometric pattern etc.). But Orhan activates commitments that refer to arithmetic concepts. For him, it is a viable way of operating in situations like these. Continuing the sequence, Orhan uses a triangle shape to draw the next element. Asked for his commitment (line 2 of the transcript above), he says that it is easier for him to use "the wall". It is one important result to show, that Orhan uses both geometric and arithmetic commitments to find the rule of the pattern and then to continue the sequence. This will play an important role in the light of the symbolically explicit use of the variable (see below). Second, it is an important result that these changes between geometric and arithmetic commitments do not occur without mathematical frictions: Although, for Orhan, it is a proper

conclusion to use a triangle shape when having determined the number of dots for the next element. But, mathematically, he does not use the given geometric structure to continue the sequence. This is an interesting result insofar, as it is typical for Orhan's learning process that he is very flexible and shows strong competences when dealing in arithmetic situations but at the same time he often acknowledges geometric commitments that are mathematically not viable. As a third result in this scene, for Orhan, the "wall" is a tool to visualize a given (or determined) number of dots. Even more: The wall, for Orhan, is *the easiest* tool to visualize in situations like these. This scene shows – on a propaedeutic level – how far geometric patterns can be *tools* for students in visualization-situations. In contrast to these results dealing with figural growing patterns, the next section will show, that – also on a preformal level – Orhan acknowledges arithmetic commitments, that refer to the structural notion of the variable.

The concept of variable in algebraic expressions between the implicit and the explicit

In a different scene, Orhan works on a given arithmetic sequence: 2, 10, 18, 26, 34,…. Orhan is first asked to determine the rule and he answers: "You always have to add 8." Using that rule, Orhan then determines the next three elements 42, 50 and 58 of the sequence. Meanwhile, the students have learned in class, that the (general) rule of a given arithmetic or geometric pattern can be made explicit with the help of an algebraic expression with a variable. Orhan writes down: "2+8x = ". His commitment can be reconstructed here: *I can make explicit generalizations of arithmetic patterns in algebraic expressions with variables*. Being asked, what the x stands for, Orhan answers: "The x means a number, let's say you want the 35^{th} element of a sequence."

Here, Orhan uses the concept of variable first to describe the arithmetic structure of the sequence and then, second, as a tool to determine certain elements of the sequence. Orhan uses the concept of *element* of a sequence, that can be determined with the help of the variable. For him, the meaning of the concept in this situation is mainly rooted in the calculation of elements with high numbers. For Orhan, the explicit concept of variable marks a specific character of a tool (c.f. Sfard & Linchevski, 1994) and he acknowledges the following commitment: *Elements of sequences can be calculated with expressions using the variable. The x stands for the element*.

The importance of this result can be seen by analysing the last scene. Orhan is asked, if 90 was an element of the sequence.

1	Orhan:	90 (…) yes
2	Int.:	(…) And why is 90 an element of the sequence?
3	Orhan:	90, eh, because 88 can be divided by 8 (88 ist in der 8er Reihe)
4	Int.:	Yes. And 90?
5	Orhan:	90 cannot be divided by 8, but 90 is part of the sequence. Then I subtracted 2 and that makes 88.

Here Orhan activates commitments, that explicitly refer to structural algebraic concepts. For Orhan, the sequence 2, 10, 18, … in this situation is a structural object with certain properties. One of these properties is that each element of the sequence can be produced by taking a number that can be divided by 8 and then add 2. In this scene, Orhan puts 2 different sequences (0, 8, 16, 24,… and 2, 10, 18, 26,…) in a certain relation and his argumentation uses the properties of both sequences. It is important to note now, that by dealing with arithmetic sequences, Orhan acknowledges commitments, that not only refer to central algebraic concepts (Kieran 2007) but that also constitute a stable concept of function (Healy & Hoyles 1999). It is a central insight here, that Orhan uses the concept of sequence not as an operational but as a structural object with certain properties, that he uses as reasons in his argumentation. Also, this scene shows that his commitments refer to an elaborated concept of equivalence and the transformation of algebraic expressions although these concepts themselves are not symbolically explicit. Still, his commitments refer to them in on a preformal level (lines 1-5 in the transcript above).

Although Orhan's symbolically explicit use of the concept of variable refers to an operational character, the detailed analysis of his commitments shows that they also refer to important structural algebraic concepts on an implicit level. His commitments, that the properties of two sequences may be compared and used within a complex argumentation, reveal the concept of equivalence, variable and equality as structural objects that are not used symbolically explicit but on a propaedeutic level. This result extends the notion of the duality of concepts within the context of generalizing to a propaedeutic dimension of conceptual use.

CONCLUSION

The empirical results presented above allow some insights into processes of concept-formation when learning the concept of variable within a learning environment, that focuses on dealing with growing figural patterns and arithmetic sequences. The inferential theoretical approach unfolds its potential especially regarding the (inferential) relation of arithmetic and geometric commitments when learning the concept of variable. The data especially shows examples of students' geometrical reasoning within generalization processes, whereas the reconstructed commitments refer to arithmetical concepts. Although we only refer to one case in this contribution, it not only shows the interplay between arithmetic and geometric commitments but also the different notions of structural and operational concept-usage on a propaedeutic as well as on a symbolically explicit level. Besides the very fine-grained analysis to study, this theoretical framework offers a tool to describe frictions and to interpret them within the individual argumentation processes that are being reconstructed with individual commitments and inferences. Finally, the data shows that this theoretical framework offers a perspective to extend and differentiate important insights (Sfard et al., 1994) regarding the symbolically explicit use of the variable to its propaedeutic usage and – more generally – it offers insights into the individual formation of concepts itself.

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